Path Integral of Bianchi I models in Loop Quantum Cosmology

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A path integral formulation of the Bianchi I models containing a massless scalar field in loop quantum cosmology is constructed. Following the strategy used in the homogenous and isotropic case, the calculation is extended to the simplest non-isotropic models according to the $\bar{\mu}$ and $\bar{\mu}'$ scheme. It is proved from the path integral angle that the quantum dynamic lacks the full invariance with respect to fiducial cell scaling in the $\bar{\mu}$ scheme, but it does not in the $\bar{\mu}'$ scheme. The investigation affirms the equivalence of the canonical approach and the path integral approach in loop quantum cosmology.

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I. INTRODUCTION

Loop quantum gravity [1] and the spinfoam formalism [2, 3] can be considered respectively as canonical quantization and covariant quantization of gravity just like what we did in ordinary quantum field theory. Applying the the techniques used in LQG to simple cosmology models, we get loop quantum cosmology [4], which is a canonical version of quantum cosmology. Naturally, we desire to understand LQC conclusions from a path integral, i.e., covariant angle. Starting from the Hilbert space of LQC, there are two kinds of path integrals. One is integral over paths in configuration space [5–8], which leads to a spinfoam like 'vertex summation' rather than a standard path integral format. The other is integral over phase space paths [9–11] from which we can get a typical continuous path integral that features continuous paths with weights given by the exponential of the phases.

In this paper, we will use phase space paths following the methods in the original work of A. Ashtekar, M. Campiglia and A. Henderson [9], and extend the LQC path integral formulation from homogenous and isotropic FRW model to the simplest homogenous Bianchi type I models [12]. Up to now there are two kinds of 'loop regularization' of the gravitational part of the scalar constraint for the Bianchi I models in the literature, referred to as $\bar{\mu}$ and $\bar{\mu}'$ scheme [13, 14]. Each of them has its own advantages and drawbacks. $\bar{\mu}$ scheme gives a difference equation in terms of affine variables and therefore the well-developed framework of the spatially flat-isotropic LQC can be straightforwardly adopted. While the $\bar{\mu}'$ scheme has better scaling properties [14], but the difference equation of this scheme is very complex. Here we use both $\bar{\mu}$ and $\bar{\mu}'$ schemes presented in [13] to perform a path integral formulation, and to explore their similarities and differences.

The organization of this paper is as follows. In the Sec. II, we present the quantum theory of Bianchi I modles in the volume and connection representations. In Sec. III, we construct the path integral formalism of quantum Bianchi I models in phase space using $\bar{\mu}$ scheme, and the construction in $\bar{\mu}'$ scheme will be presented in the Sec. IV. Finally, we discuss the results obtained from those two schemes in the Sec. V.

II. LQC OF THE DIAGONAL BIANCHI I MODELS

Bianchi cosmologies are homogeneous cosmological models, in which there is a foliation of spacetime $M = \Sigma \times \mathbb{R}$ such that Σ is space-like and there is a transitive isometry group freely acting on Σ. Thanks to these symmetry properties, in classical theory, after the diagonalizing and rescaling processes of the Ashtekar variables (see [12]), the only non-vanishing constraint(scalar constraint) for the Bianchi I cosmology (coupled with a massless scalar field ϕ) is given by

$$C = -\frac{1}{8\pi G\gamma^2}\left(c^1 p_1 c^2 p_2 + c^1 p_1 c^3 p_3 + c^2 p_2 c^3 p_3\right) + \frac{1}{2}p_\phi^2, \tag{1}$$

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where p_ϕ is the momentum of the scalar field, while c^i and p_j are the reduced and rescaled Ashtekar variables. The Poisson brackets of c^i and p_j are

$$\{c^i, p_j\} = 8\pi\gamma\, G\, \delta^i_j. \tag{2}$$

In the isotropic models, there is a reflection symmetry $\Pi(p) = -p$, which corresponds to the orientation reversal of triads. These are gouge transformations which leave the Hamiltonian constraint invariant. In Bianchi I case, we have three reflections Π_i's corresponding respectively to the reversal of one of the triad while leaving the other two fixed. The Hamiltonian flow is invariant under the action of each Π_i [14]. Hence it suffices to restrict our attention to one of the octant of three p_i's. In this paper, we focus on the positive octant in which all three p_i's are positive.

A. $\bar{\mu}$ scheme

In the canonical quantization theory, i.e., loop quantum cosmology, there exists a comprehensive construction of the operator corresponding to the Eq. (1) in the full Loop quantum gravity scheme [12]. The full LQG quantization proposes two kinds of quantum corrections to the scalar constraint, one of which is the inverse triad correction by using a so-called Thiemann trick, while the other one is holonomy correction which means to correct the curvature term by using a $SU(2)$ holonomy. In this paper, we start out setting the lapse function $N = |p_1 p_2 p_3|^{1/2}$ in the classical theory as in [14]. This treatment doesn't only make the Hamiltonian constraint simpler, but also avoids the use of inverse triad correction. Hence, we can use the regularized constraint [13] to replace Eq. (1), in which only the LQG effects from holonomy correction are taken into account:

$$C = -\frac{1}{8\pi G\gamma^2}\left[\frac{\sin\left(\bar{\mu}_1 c^1\right)}{\bar{\mu}_1}p_1\frac{\sin\left(\bar{\mu}_2 c^2\right)}{\bar{\mu}_2}p_2 + \frac{\sin\left(\bar{\mu}_1 c^1\right)}{\bar{\mu}_1}p_1\frac{\sin\left(\bar{\mu}_3 c^3\right)}{\bar{\mu}_3}p_3 + \frac{\sin\left(\bar{\mu}_2 c^2\right)}{\bar{\mu}_2}p_2\frac{\sin\left(\bar{\mu}_3 c^3\right)}{\bar{\mu}_3}p_3\right] + \frac{p_\phi^2}{2} = 0. \tag{3}$$

In $\bar{\mu}$ scheme, $\bar{\mu}_i = \sqrt{\Delta/|p_i|}$, where $\Delta = 4\sqrt{3}\pi\gamma l_{Pl}^2$ is the 'area gap' [15]. Usually, we can make an algebraic simplification by introducing new phase space variables:

$$\nu_i = \frac{{p_i}^{3/2}}{6\pi G}, b_i = \frac{c_i}{\gamma\sqrt{p_i}}, \tag{4}$$

where the index i does not sum over and their Poisson bracket is given by

$$\{b^i, \nu_j\} = 2\delta^i_j. \tag{5}$$

Then the kinematical Hilbert space of the gravitational part is given by

$$\mathcal{H}^{grav}_{Kin} = L^2(\mathbb{R}_{Bohr}, d\mu_{Bohr})^3, \tag{6}$$

with the orthonormal basis $|\nu_1, \nu_2, \nu_3\rangle = |\nu_1\rangle\,|\nu_2\rangle\,|\nu_3\rangle$ satisfying

$$\langle \nu_1, \nu_2, \nu_3\,|\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3\rangle = \delta_{\nu_1,\tilde{\nu}_1}\delta_{\nu_2,\tilde{\nu}_2}\delta_{\nu_3,\tilde{\nu}_3}. \tag{7}$$

Any state $|\Psi(\nu_1, \nu_2, \nu_3)\rangle \in \mathcal{H}^{grav}_{Kin}$ can be decomposed in the orthonormal basis as

$$|\Psi(\nu_1, \nu_2, \nu_3)\rangle = \sum_{\nu_1,\nu_2,\nu_3} \Psi(\nu_1, \nu_2, \nu_3)\,|\nu_1, \nu_2, \nu_3\rangle\,. \tag{8}$$

To define the constraint operator, we have to define two kinds of operators first. One is the volume operator defined as

$$\hat{V}\,|\nu_1, \nu_2, \nu_3\rangle = 6\pi G\gamma\sqrt{\Delta}|\nu_1\nu_2\nu_3|^{1/3}\,|\nu_1, \nu_2, \nu_3\rangle\,, \tag{9}$$

where $\hat{V} = |\hat{V}_1\hat{V}_2\hat{V}_3|^{1/3}$, and $\hat{V}_i\,|\nu_1, \nu_2, \nu_3\rangle = 6\pi G\gamma\sqrt{\Delta}\nu_i\,|\nu_1, \nu_2, \nu_3\rangle$, which is the operator corresponding to p_i. The other is the unitary shift operator

$$\hat{U}^i|\nu_i\rangle = \exp\widehat{(i\bar{\mu}_i c^i)}\,|\nu_i\rangle = |\nu_i + 2\ell_0\hbar\rangle\,, \tag{10}$$

where ℓ_0 is related to the 'area gap' by $\ell_0^2 = \gamma^2\Delta$, where γ is the Barbero-Immirzi parameter. Replace the function $\sin\bar{\mu}_i c^i$ in Eq. (3) by the operator $\widehat{\sin\bar{\mu}_i c^i}$ using the unitary shift operator defined above, we get

$$\begin{aligned} \widehat{\sin(\bar{\mu}_i c^i)}\,|\nu_i\rangle &= \frac{1}{2i}\left(\widehat{\exp(i\bar{\mu}_i c^i)} - \widehat{\exp(-i\bar{\mu}_i c^i)}\right)|\nu_i\rangle \\ &= \frac{1}{2i}\left(|\nu_i + 2\ell_0\hbar\rangle - |\nu_i - 2\ell_0\hbar\rangle\right). \end{aligned} \tag{11}$$

Then we fix the factor ordering ambiguity of Eq. (3), by defining the operator

$$\begin{aligned} \hat{\Theta}_i|\nu_i\rangle :=& \frac{1}{2}\left(\frac{\widehat{\sin\bar{\mu}_i c^i}}{\sqrt{\Delta}}\hat{V}_i + \hat{V}_i\frac{\widehat{\sin\bar{\mu}_i c^i}}{\sqrt{\Delta}}\right)|\nu_i\rangle \\ =& -3i\pi G\gamma\left[(\nu_i + \ell_0\hbar)\,|\nu_i + 2\ell_0\hbar\rangle - (\nu_i - \ell_0\hbar)\,|\nu_i - 2\ell_0\hbar\rangle\right]. \end{aligned} \tag{12}$$

Notice that $[\hat{\Theta}_i, \hat{\Theta}_j]|\nu_1, \nu_2, \nu_3\rangle = 0$ for $i \neq j$. The gravitational part of the scalar constraint operator can be written in terms of these operator as

$$\hat{C}^{grav} = -\frac{1}{8\pi G\gamma^2}\left(\hat{\Theta}_1\hat{\Theta}_2 + \hat{\Theta}_2\hat{\Theta}_3 + \hat{\Theta}_1\hat{\Theta}_3\right). \tag{13}$$

The action of this operator on arbitrary basis element of the kinematical Hilbert space is [16]

$$\begin{aligned} &\hat{C}^{grav}|\nu_1, \nu_2, \nu_3\rangle \\ =\; &\frac{9\pi G}{8}\left[(\nu_1+1)(\nu_2+1)\,|\nu_1+2, \nu_2+2, \nu_3\rangle - (\nu_1-1)(\nu_2+1)\,|\nu_1-2, \nu_2+2, \nu_3\rangle\right. \\ &-(\nu_1+1)(\nu_2-1)\,|\nu_1+2, \nu_2-2, \nu_3\rangle + (\nu_1-1)(\nu_2-1)\,|\nu_1-2, \nu_2-2, \nu_3\rangle \\ &+(\nu_2+1)(\nu_3+1)\,|\nu_1, \nu_2+2, \nu_3+2\rangle - (\nu_2-1)(\nu_3+1)\,|\nu_1, \nu_2-2, \nu_3+2\rangle \\ &-(\nu_2+1)(\nu_3-1)\,|\nu_1, \nu_2+2, \nu_3-2\rangle + (\nu_2-1)(\nu_3-1)\,|\nu_1, \nu_2-2, \nu_3-2\rangle \\ &+(\nu_1+1)(\nu_3+1)\,|\nu_1+2, \nu_2, \nu_3+2\rangle - (\nu_1-1)(\nu_3+1)\,|\nu_1-2, \nu_2, \nu_3+2\rangle \\ &\left.-(\nu_1+1)(\nu_3-1)\,|\nu_1+2, \nu_2, \nu_3-2\rangle + (\nu_1-1)(\nu_3-1)\,|\nu_1-2, \nu_2, \nu_3-2\rangle\right], \end{aligned} \tag{14}$$

where the $\ell_0\hbar$ terms are omitted for short. From this equation we can see that in the $\bar{\mu}$ scheme, the space of solutions to the quantum constraint is very similar to the isotropic case. ν_1, ν_2, ν_3 are discrete variables and they are supported on a specific superselections respectively. They can take the values of $\nu_i = (\varepsilon_i + 2n_i)\ell_0\hbar$, where the parameter $\varepsilon_i \in [0, 1]$, and $n_i \in \mathbb{N}$. As mentioned previously, we focus on the positive octant here. This precondition simplify the calculation but do not affect the physical meaning. If we consider the general situation, a factor $sgn(p_i)$ will appear in the expression of $\hat{\Theta}_i$. And as a function on phase space, it does not commute with $\sin\bar{\mu}_i c^i$, hence their product as operators is not symmetric. It is necessary to realize this point, although our simplification makes its role less important. For the details about this situation, see [17].

B. $\bar{\mu}'$ scheme

We still use the regularized constraint Eq. (3) in $\bar{\mu}'$ scheme. Only $\bar{\mu}_i$ changes to $\bar{\mu}'_i$ in this situation, and $\bar{\mu}'_i$ takes the form

$$\bar{\mu}'_1 = \sqrt{\frac{|p_1|\Delta}{|p_2 p_3|}}, \quad \bar{\mu}'_2 = \sqrt{\frac{|p_2|\Delta}{|p_1 p_3|}}, \quad \bar{\mu}'_3 = \sqrt{\frac{|p_3|\Delta}{|p_1 p_2|}}. \tag{15}$$

Owning to this condition, it's inconvenient to still use ν_i as configuration variables. By introducing new variables λ_i,

$$\lambda_i = \frac{\sqrt{p_i}}{(4\pi G)^{1/3}}, \tag{16}$$

the algebra of the $\bar{\mu}'$ scheme could be much more simplified [14]. The variable conjugate to λ_i is

$$k^i = \frac{2\sqrt{p_i}c^i}{\gamma(4\pi G)^{2/3}}. \tag{17}$$

We can prove that the resulting Poisson bracket between k^i and λ_j is

$$\{k^i, \lambda_j\} = 2\delta^i_j. \tag{18}$$

Then the new orthonormal basis is $|\lambda_1, \lambda_2, \lambda_3\rangle$ in $\mathcal{H}^{grav}_{Kin}$, and the fundamental volume operator and unitary shift are given by

$$\hat{V}\,|\lambda_1, \lambda_2, \lambda_3\rangle = 2\pi G\gamma\sqrt{\Delta}v\,|\lambda_1, \lambda_2, \lambda_3\rangle\,, \tag{19}$$

and

$$\hat{E}_i|\lambda_i\rangle = \exp\left(\widehat{i\bar{\mu}_i c^i}\right)|\lambda_i\rangle = \left|\lambda_i + \frac{\ell_0\hbar}{\lambda_j\lambda_k}\right\rangle. \tag{20}$$

Here, $v = 2\lambda_1\lambda_2\lambda_3$ is the configuration variable related to the volume of the elementary cell, and the factor 2 ensures that this v reduces to the v used in the isotropic case. We can also use λ_i, λ_j, and v as the basic configuration variables to substitute for λ_1, λ_2, and λ_3. The indices $i, j, k = 1, 2, 3$ appeared in Eq. (19) are required to be different with each other, and, as usual, the index i does not sum over. Having these operators in hands, we could get the gravitational part of the scalar constraint operator. Its action on arbitrary basis element of the new kinematical Hilbert space is [14]

$$\begin{aligned}\hat{C}^{grav}\,|\lambda_1, \lambda_2, v\rangle \;=\; & \frac{\pi G}{2}\sqrt{v}\Big[(v+2)\sqrt{v+4}\,|\lambda_1, \lambda_2, v\rangle^+_4 - (v+2)\sqrt{v}\,|\lambda_1, \lambda_2, v\rangle^+_0 \\ & -(v-2)\sqrt{v}|\lambda_1, \lambda_2, v\rangle^-_0 + (v-2)\sqrt{|v-4|}|\lambda_1, \lambda_2, v\rangle^-_4\Big]\,,\end{aligned} \tag{21}$$

where

$$\begin{aligned}|\lambda_1, \lambda_2, v\rangle^\pm_4 \;=\; & \left|\frac{v\pm4}{v\pm2}\cdot\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\pm4\right\rangle + \left|\frac{v\pm4}{v\pm2}\cdot\lambda_1, \lambda_2, v\pm4\right\rangle \\ & + \left|\frac{v\pm2}{v}\cdot\lambda_1, \frac{v\pm4}{v\pm2}\cdot\lambda_2, v\pm4\right\rangle + \left|\frac{v\pm2}{v}\cdot\lambda_1, \lambda_2, v\pm4\right\rangle \\ & + \left|\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\pm4\right\rangle + \left|\lambda_1, \frac{v\pm4}{v\pm2}\cdot\lambda_2, v\pm4\right\rangle,\end{aligned} \tag{22}$$

and

$$\begin{aligned}|\lambda_1, \lambda_2, v\rangle^\pm_0 \;=\; & \left|\frac{v\pm2}{v}\cdot\lambda_1, \frac{v}{v\pm2}\cdot\lambda_2, v\right\rangle + \left|\frac{v\pm2}{v}\cdot\lambda_1, \lambda_2, v\right\rangle \\ & + \left|\frac{v}{v\pm2}\cdot\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\right\rangle + \left|\frac{v}{v\pm2}\cdot\lambda_1, \lambda_2, v\right\rangle \\ & + \left|\lambda_1, \frac{v}{v\pm2}\cdot\lambda_2, v\right\rangle + \left|\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\right\rangle.\end{aligned} \tag{23}$$

As what we did in Eq. (14), the $\ell_0\hbar$ terms are omitted here. From Eq. (22) and Eq. (23), we can see that the variable v is fine, and the wave function only involves $(v - 4\ell_0\hbar)\,, v, (v + 4\ell_0\hbar)$ terms. It is exactly the same as in the isotropic case. On the contrary, the situation for λ_1, λ_2 is much different. We see that they depend only on the value of v. This dependence is through fractional factors whose denominator is two or four units bigger or smaller than the numerator. This lead to the consequence that the iterative action of the constraint operator derives only to states whose quantum numbers λ_i are of the form $\lambda_i = \omega_\epsilon\lambda'_i$, here λ'_i is the initial value of λ_i and ϵ is a constant number that $v = \epsilon + 4n\ell_0\hbar, n \in \mathbb{N}$. ω_ϵ belonging to the set [18]

$$\Omega_\epsilon = \left\{\left(\frac{\epsilon-2}{\epsilon}\right)^z \prod_{m,n\in\mathbb{N}}\left(\frac{\epsilon+2m}{\epsilon+2n}\right)^{k^m_n}\right\}, \tag{24}$$

where $k^m_n \in \mathbb{N}$, and $z \in \mathbb{Z}$ if $\epsilon > 2$, while $z = 0$ when $\epsilon \leq 2$. The discrete set Ω_ϵ is countably infinite and turns out to be dense in the positive real line. The proof of this statement see Appendix D3 of [18]. So, On the one hand, λ_i is superselected in separable sectors, have countable numbers of values. On the other hand, the dependence on the fractional factors makes it can take values in entire positive real line.

III. PHASE SPACE PATH INTEGRAL IN $\bar{\mu}$ SCHEME

In this section, we calculate the extraction amplitude $A(\vec{\nu}_f,\phi_f;\vec{\nu}_i,\phi_i)$ for Bianchi I cosmology (here $\vec{\nu}$ is short for $\{\nu_1,\nu_2,\nu_3\}$). As we know, in the 'timeless' framework of LQC, the whole information of the quantum dynamics is encoded in $A(\vec{\nu}_f,\phi_f;\vec{\nu}_i,\phi_i)$, which is the transition amplitude for our phase space path integral just as in ordinary quantum mechanics. We can express it as [5]

$$A(\vec{\nu}_f,\phi_f;\vec{\nu}_i,\phi_i)=\int d\alpha\,\left\langle\vec{\nu}_f,\phi_f\left|e^{\frac{i}{\hbar}\alpha\hat{C}}\right|\vec{\nu}_i,\phi_i\right\rangle, \tag{25}$$

where the total constraint operator $\hat{C}$ is composed of two parts: the gravitational part and the scalar field part

$$\hat{C}=\hat{C}^{grav}+\hat{C}^{matt}. \tag{26}$$

The two operators $\hat{C}^{grav}$ and $\hat{C}^{matt}$ are commutative, so they can act on their own Hilbert space, $\mathcal{H}^{grav}_{Kin}$ and $\mathcal{H}^{matt}_{Kin}$, respectively. Then following the spirit of [9], decompose the fictitious evolution into N evolutions of length $\epsilon=1/N$, and insert complete basis in between each factor, we get

$$\left\langle\vec{\nu}_f,\phi_f\left|e^{\frac{i}{\hbar}\alpha\hat{C}}\right|\vec{\nu}_i,\phi_i\right\rangle=\sum_{\vec{\nu}_{N-1}\cdots\vec{\nu}_1}\int d\phi_{N-1}\cdots d\phi_1\left\langle\vec{\nu}_N,\phi_N\left|e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}}\right|\vec{\nu}_{N-1},\phi_{N-1}\right\rangle\cdots\left\langle\vec{\nu}_1,\phi_1\left|e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}}\right|\vec{\nu}_0,\phi_0\right\rangle, \tag{27}$$

where $\langle\vec{\nu}_N,\phi_N|\equiv\langle\vec{\nu}_f,\phi_f|$ and $|\vec{\nu}_0,\phi_0\rangle\equiv|\vec{\nu}_i,\phi_i\rangle$.

Firstly, we need to calculate the n-th term appearing in above expression. Using the Eq. (26), it can be stated as

$$\begin{aligned}&\left\langle\nu^1_{n+1},\nu^2_{n+1},\nu^3_{n+1},\phi_{n+1}\left|e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}}\right|\nu^1_n,\nu^2_n,\nu^3_n,\phi_n\right\rangle\\ =\;&\left\langle\phi_{n+1}\left|e^{\frac{i}{\hbar}\varepsilon\alpha\hat{C}^{matt}}\right|\phi_n\right\rangle\left\langle\nu^1_{n+1},\nu^2_{n+1},\nu^3_{n+1}\left|e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}^{grav}}\right|\nu^1_n,\nu^2_n,\nu^3_n\right\rangle.\end{aligned} \tag{28}$$

As usual, the matter Hilbert space is the standard one, $\mathcal{H}^{matt}_{Kin}=L^2(\mathbb{R},d\phi)$. By an ordinary quantum mechanics like calculation, the scalar field factor is

$$\left\langle\phi_{n+1}\left|e^{\frac{i}{\hbar}\varepsilon\alpha\hat{C}^{matt}}\right|\phi_n\right\rangle=\left\langle\phi_{n+1}\left|e^{\frac{i}{\hbar}\varepsilon\alpha\hat{p}^2}\right|\phi_n\right\rangle=\int\frac{dp_n}{2\pi}e^{\frac{i}{\hbar}p_n(\phi_{n+1}-\phi_n)+\frac{i}{\hbar}\epsilon\alpha p_n^2}, \tag{29}$$

and the gravitational factor is

$$\begin{aligned}&\left\langle\nu^1_{n+1},\nu^2_{n+1},\nu^3_{n+1}\left|e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}^{grav}}\right|\nu^1_n,\nu^2_n,\nu^3_n\right\rangle\\ =\;&\delta_{\nu^1_{n+1},\nu^1_n}\delta_{\nu^2_{n+1},\nu^2_n}\delta_{\nu^3_{n+1},\nu^3_n}+\frac{i}{\hbar}\epsilon\alpha\left\langle\nu^1_{n+1},\nu^2_{n+1},\nu^3_{n+1}\left|\hat{C}^{grav}\right|\nu^1_n,\nu^2_n,\nu^3_n\right\rangle+\mathcal{O}(\epsilon^2),\end{aligned} \tag{30}$$

where we made an expansion in ϵ, and $\mathcal{O}(\epsilon^2)$ is the high order terms of the expansion. Using Eq. (3), the matrix element of $\hat{C}^{grav}$ is

$$\begin{aligned}&\left\langle\nu^1_{n+1},\nu^2_{n+1},\nu^3_{n+1}\left|\hat{C}^{grav}\right|\nu^1_n,\nu^2_n,\nu^3_n\right\rangle\\ =\;&\frac{9\pi G}{8}\Bigg[\frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^2_{n+1}+\nu^2_n}{2}(\delta_{\nu^1_{n+1},\nu^1_n+2}\delta_{\nu^2_{n+1},\nu^2_n+2}\delta_{\nu^3_{n+1},\nu^3_n}-\delta_{\nu^1_{n+1},\nu^1_n-2}\delta_{\nu^2_{n+1},\nu^2_n+2}\delta_{\nu^3_{n+1},\nu^3_n}\\ &-\delta_{\nu^1_{n+1},\nu^1_n+2}\delta_{\nu^2_{n+1},\nu^2_n-2}\delta_{\nu^3_{n+1},\nu^3_n}+\delta_{\nu^1_{n+1},\nu^1_n-2}\delta_{\nu^2_{n+1},\nu^2_n-2}\delta_{\nu^3_{n+1},\nu^3_n}\Big)\\ &+\frac{\nu^2_{n+1}+\nu^2_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\Big(\delta_{\nu^2_{n+1},\nu^2_n+2}\delta_{\nu^3_{n+1},\nu^3_n+2}\delta_{\nu^1_{n+1},\nu^1_n}-\delta_{\nu^2_{n+1},\nu^2_n-2}\delta_{\nu^3_{n+1},\nu^3_n+2}\delta_{\nu^1_{n+1},\nu^1_n}\\ &-\delta_{\nu^2_{n+1},\nu^2_n+2}\delta_{\nu^3_{n+1},\nu^3_n-2}\delta_{\nu^1_{n+1},\nu^1_n}+\delta_{\nu^2_{n+1},\nu^2_n-2}\delta_{\nu^3_{n+1},\nu^3_n-2}\delta_{\nu^1_{n+1},\nu^1_n}\Big)\\ &+\frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\Big(\delta_{\nu^1_{n+1},\nu^1_n+2}\delta_{\nu^3_{n+1},\nu^3_n+2}\delta_{\nu^2_{n+1},\nu^2_n}-\delta_{\nu^1_{n+1},\nu^1_n-2}\delta_{\nu^3_{n+1},\nu^3_n+2}\delta_{\nu^2_{n+1},\nu^2_n}\\ &-\delta_{\nu^1_{n+1},\nu^1_n+2}\delta_{\nu^3_{n+1},\nu^3_n-2}\delta_{\nu^2_{n+1},\nu^2_n}+\delta_{\nu^1_{n+1},\nu^1_n-2}\delta_{\nu^3_{n+1},\nu^3_n-2}\delta_{\nu^2_{n+1},\nu^2_n}\Big)\Bigg].\end{aligned} \tag{31}$$

As demonstrated before, we choose a specific superselection of ν_i such that $\nu_i = 2n_i\ell_0\hbar$, where $n_i \in \mathbb{N}$. Then we can use the Fourier expansion formula of the Kronecker delta

$$\delta\nu'\nu = \frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} dbe^{-ib\left(\nu'-\nu\right)/2\hbar}, \tag{32}$$

where, obviously, b is the conjugate variable to ν and take values in the range $(0, \pi/\ell_0)$. By using Eq. (32), then Eq. (31) can be expressed as

$$\begin{aligned}
&\left\langle\nu^1_{n+1}, \nu^2_{n+1}, \nu^3_{n+1}\left|e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}^{grav}}\right|\nu^1_n, \nu^2_n, \nu^3_n\right\rangle \\
= &\left(\frac{\ell_0}{\pi}\right)^3 \int d\vec{b}_{n+1} e^{-\frac{i}{\hbar}\frac{b^1_{n+1}(\nu^1_{n+1}-\nu^1_n)}{2}} \cdot e^{-\frac{i}{\hbar}\frac{b^2_{n+1}(\nu^2_{n+1}-\nu^2_n)}{2}} \cdot e^{-\frac{i}{\hbar}\frac{b^3_{n+1}(\nu^3_{n+1}-\nu^3_n)}{2}} \\
&\times\left\{1 - \frac{i}{\hbar}\epsilon\alpha\frac{9}{2}\pi G\left[\frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^2_{n+1}+\nu^2_n}{2}\sin\ell_0 b^1_{n+1}\sin\ell_0 b^2_{n+1}\right.\right. \\
&\left.\left.+\frac{\nu^2_{n+1}+\nu^2_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\sin\ell_0 b^2_{n+1}\sin\ell_0 b^3_{n+1} + \frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\sin\ell_0 b^1_{n+1}\sin\ell_0 b^3_{n+1} + \mathcal{O}(\epsilon^2)\right]\right\}.
\end{aligned} \tag{33}$$

Now according to the calculation before, we could put the scalar field part and gravitational part together, and Eq. (27) takes the form

$$\begin{aligned}
&\left\langle\vec{\nu}_f, \phi_f\left|e^{i\alpha\hat{C}}\right|\vec{\nu}_i, \phi_i\right\rangle \\
= &\sum_{\nu_{N-1}\cdots\nu_1}\left(\frac{\ell_0}{\pi}\right)^{3N}\int d\vec{b}_N\cdots d\vec{b}_1\cdot\left(\frac{1}{2\pi}\right)^{3N}\int dp_N\cdots dp_1 e^{\frac{i}{\hbar}S_N} + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
S_N = &\ \epsilon\sum_{n=0}^{N-1}\left(p_{n+1}\frac{\phi_{n+1}-\phi_n}{\epsilon} - \frac{b^1_{n+1}}{2}\frac{\nu^1_{n+1}-\nu^1_n}{\epsilon} - \frac{b^2_{n+1}}{2}\frac{\nu^2_{n+1}-\nu^2_n}{\epsilon} - \frac{b^3_{n+1}}{2}\frac{\nu^3_{n+1}-\nu^3_n}{\epsilon}\right) \\
&+\alpha\left[p_n^2 - \frac{9}{2}\pi G\left(\frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^2_{n+1}+\nu^2_n}{2}\sin\ell_0 b^1_{n+1}\sin\ell_0 b^2_{n+1}\right.\right. \\
&\left.\left.+\frac{\nu^2_{n+1}+\nu^2_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\sin\ell_0 b^2_{n+1}\sin\ell_0 b^3_{n+1} + \frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\sin\ell_0 b^1_{n+1}\sin\ell_0 b^3_{n+1}\right)\right].
\end{aligned} \tag{35}$$

This is the 'discrete-time action' of the Bianchi I cosmology. Just like in [9], the final step is to take the limit $N \to \infty$, and because the variable ν_i is discrete, it is impossible to interpret the $(\nu^i_{n+1} - \nu^i_n)/\epsilon$ as a derivative, we should transform the terms $(\nu^i_{n+1} - \nu^i_n)/\epsilon$ into $(b^i_{n+1} - b^i_n)/\epsilon$:

$$\frac{1}{2}\left(\nu_N b_N - \nu_0 b_1\right) = \frac{1}{2}\sum_{n=0}^{N-1}\left(b_{n+1}\nu_{n+1} - \nu_n b_n\right). \tag{36}$$

Then

$$\epsilon\sum_{n=0}^{N-1}\left[-\frac{b_{n+1}}{2}\frac{\nu_{n+1}-\nu_n}{\epsilon}\right] = \epsilon\sum_{n=0}^{N-1}\left[\frac{\nu_n}{2}\frac{b_{n+1}-b_n}{\epsilon}\right] + \frac{1}{2}(b_1\nu_0 - b_N\nu_N). \tag{37}$$

Using this transformation, S_N can be rewritten as

$$\begin{aligned}
S_N = &\ \epsilon\left\{p_{n+1}\frac{\phi_{n+1}-\phi_n}{\epsilon} + \frac{\nu^1_n}{2}\frac{b^1_{n+1}-b^1_n}{\epsilon} + \frac{\nu^2_n}{2}\frac{b^2_{n+1}-b^2_n}{\epsilon} + \frac{\nu^3_n}{2}\frac{b^3_{n+1}-b^3_n}{\epsilon}\right. \\
&+\alpha\left[p_n^2 - \frac{9}{2}\pi G\left(\frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^2_{n+1}+\nu^2_n}{2}\sin\ell_0 b^1_{n+1}\sin\ell_0 b^2_{n+1}\right.\right. \\
&\left.\left.\left.+\frac{\nu^2_{n+1}+\nu^2_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\sin\ell_0 b^2_{n+1}\sin\ell_0 b^3_{n+1} + \frac{\nu^1_{n+1}+\nu^1_n}{2}\frac{\nu^3_{n+1}+\nu^3_n}{2}\sin\ell_0 b^1_{n+1}\sin\ell_0 b^3_{n+1}\right)\right]\right\} \\
&-\frac{1}{2}(\vec{\nu}_f\vec{b}_f - \vec{\nu}_i\vec{b}_i).
\end{aligned} \tag{38}$$

Finally, take the limit $N \to \infty$, we have

$$A(\vec{\nu}_f, \phi_f; \vec{\nu}_i, \phi_i) = \int \alpha \int [D\nu_q(\tau)][Db_q(\tau)][Dp(\tau)][D\phi(\tau)] e^{\frac{i}{\hbar}\overline{S}}, \tag{39}$$

where

$$\begin{aligned} \overline{S} &= \int_0^1 d\tau \left\{ p\dot{\phi} + \frac{1}{2}\vec{\nu}\cdot\dot{\vec{b}} - \alpha \left[p^2 - \frac{9}{2}\pi G(\nu^1\nu^2 \sin\ell_0 b^1 \sin\ell_0 b^2 + \nu^2\nu^3 \sin\ell_0 b^2 \sin\ell_0 b^3 \right.\right. \\ &\quad \left.\left. +\nu^1\nu^3 \sin\ell_0 b^1 \sin\ell_0 b^3)\right] - \frac{1}{2}(\vec{\nu}_f\cdot\vec{b}_f - \vec{\nu}_i\cdot\vec{b}_i) \right\}. \end{aligned} \tag{40}$$

As in the isotropic case, we use the subscript q here to emphasize that now we can take a sum in the geometrical sector including only the 'quantum paths'. Also, we use the same trick to transform it to the familiar format

$$\frac{\pi}{\ell_0}\sum_{\nu_n}\int_0^{\pi/\ell_0} db_n \to \int_{-\infty}^{\infty} d\nu_n \int_{-\infty}^{\infty} db_n. \tag{41}$$

Using Eq. (41), we get the final expression of the extraction amplitude

$$A(\vec{\nu}_f, \phi_f; \vec{\nu}_i, \phi_i) = \int d\alpha \int [D\nu(\tau)][Db(\tau)][Dp(\tau)][D\phi(\tau)] e^{\frac{i}{\hbar}S}, \tag{42}$$

where

$$\begin{aligned} S &= \int_0^1 d\tau \left\{ p\dot{\phi} - \frac{1}{2}\vec{b}\cdot\dot{\vec{\nu}} - \alpha \left[p^2 \right.\right. \\ &\quad -\frac{9}{2}\pi G(\nu^1\nu^2 \sin\ell_0 b^1 \sin\ell_0 b^2 + \nu^2\nu^3 \sin\ell_0 b^2 \sin\ell_0 b^3 \\ &\quad \left.\left. +\nu^1\nu^3 \sin\ell_0 b^1 \sin\ell_0 b^3)\right] \right\}. \end{aligned} \tag{43}$$

This is the path integral formulation and its action we desired. Because we used Eq. (41), the integration about ν_i and b_i are taken from $-\infty$ to ∞, with the boundary terms in Eq. (40) canceling each other out. And then we are able to integrate over all paths in the classical phase space as in usual path integrals.

IV. PHASE SPACE PATH INTEGRAL IN $\bar{\mu}'$ SCHEME

The construction of a phase space path integral in $\bar{\mu}'$ scheme is more complex and subtler than in $\bar{\mu}$ scheme. As we illustrated in II B, the new orthonormal basis in $\mathcal{H}^{grav}_{Kin}$ can be $|\lambda_1, \lambda_2, \lambda_3\rangle$ or $|\lambda_i, \lambda_j, v\rangle$. In order to compare with the $\bar{\mu}$ scheme situation conveniently, we choose the former one. Following the procedure employed in III, we are going to calculate the extraction amplitude $A\left(\vec{\lambda}_f, \phi_f; \vec{\lambda}_i, \phi_i\right)$.

The matter part of the extraction amplitude we use here is the same as in $\bar{\mu}$ scheme, so we ignore it for a little while and directly use it at the end. Decompose the fictitious evolution into N evolutions of length $\epsilon = \frac{1}{N}$, the n-th term of the gravitational part is

$$\begin{aligned} \left\langle \vec{\lambda}^{n+1} \left| e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}^{grav}} \right| \vec{\lambda}^{n} \right\rangle &= \delta(\lambda_1^{n+1} - \lambda_1^n)\delta(\lambda_2^{n+1} - \lambda_2^n)\delta(\lambda_3^{n+1} - \lambda_3^n) \\ &\quad + \frac{i}{\hbar}\epsilon\alpha\langle\lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1}|\hat{C}^{grav}|\lambda_1^n, \lambda_2^n, \lambda_3^n\rangle + \mathcal{O}(\epsilon^2). \end{aligned} \tag{44}$$

As we known, although the number of possible values of λ_i is countable, there does not exist an obvious superselection with respect to it, therefore, we have to work with the entire positive real line it spans. So, for the possibility to perform a path integral, in a sense, we have used an approximation here. The Dirac delta is used rather than Kronecker delta to express the inner product $\left\langle \lambda_i^{n+1} \, |\lambda_i^n \right\rangle$, which means we will take λ_i as continuous variables below. This assumption may generate a little confusion. When we think λ_i is continuous, the quantum theory will be changed which caused by the changing of the Hilbert space. But, what we interest in the following is the effective action, which is a semiclassical quantity itself. And consider the properties of λ_i, it is reasonable to perform a semiclassical approximation that let

λ_i to be continuous in the following calculations and do not affect the quantum theory. It is worth to emphasize that we will focus on the semiclassical situation from now on.

Then, make a algebraic transformation: $v' = v + 2$, $\lambda'_i = v'\lambda_i/(v'-2)$; $v'' = v - 2$, $\lambda''_i = v''\lambda_i/(v''+2)$. In term of Eqs. (21), (22) and (23), the matrix element of $\hat{C}^{grav}$ is expressed as

$$\left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \hat{C}^{grav} \right| \lambda_1^n, \lambda_2^n, \lambda_3^n \right\rangle = \frac{\pi G}{2} \sqrt{v^n v^{n+1}} (v' \langle A \rangle + v'' \langle B \rangle), \tag{45}$$

where

$$\begin{aligned} \langle A \rangle \;=\; & \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n \pm \frac{\ell_0 \hbar}{\lambda_2'^n \lambda_3^n}, \lambda_2'^n, \lambda_3^n \right.\right\rangle \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n \pm \frac{\ell_0 \hbar}{\lambda_2^n \lambda_3'^n}, \lambda_2^n, \lambda_3'^n \right.\right\rangle \\ & \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n, \lambda_2^n \pm \frac{\ell_0 \hbar}{\lambda_1^n \lambda_3'^n}, \lambda_3'^n \right.\right\rangle \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1'^n, \lambda_2^n \pm \frac{\ell_0 \hbar}{\lambda_1'^n \lambda_3^n}, \lambda_3^n \right.\right\rangle \\ & \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1'^n, \lambda_2^n, \lambda_3^n \pm \frac{\ell_0 \hbar}{\lambda_1'^n \lambda_2^n} \right.\right\rangle \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n, \lambda_2'^n, \lambda_3^n \pm \frac{\ell_0 \hbar}{\lambda_1^n \lambda_2'^n} \right.\right\rangle, \end{aligned} \tag{46}$$

and

$$\begin{aligned} \langle B \rangle \;=\; & \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n \mp \frac{\ell_0 \hbar}{\lambda_2''^n \lambda_3^n}, \lambda_2''^n, \lambda_3^n \right.\right\rangle \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n \mp \frac{\ell_0 \hbar}{\lambda_2^n \lambda_3''^n}, \lambda_2^n, \lambda_3''^n \right.\right\rangle \\ & \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n, \lambda_2^n \mp \frac{\ell_0 \hbar}{\lambda_1^n \lambda_3''^n}, \lambda_3''^n \right.\right\rangle \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1''^n, \lambda_2^n \mp \frac{\ell_0 \hbar}{\lambda_1''^n \lambda_3^n}, \lambda_3^n \right.\right\rangle \\ & \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1''^n, \lambda_2^n, \lambda_3^n \mp \frac{\ell_0 \hbar}{\lambda_1''^n \lambda_2^n} \right.\right\rangle \pm \left\langle \lambda_1^{n+1}, \lambda_2^{n+1}, \lambda_3^{n+1} \left| \lambda_1^n, \lambda_2''^n, \lambda_3^n \mp \frac{\ell_0 \hbar}{\lambda_1^n \lambda_2''^n} \right.\right\rangle. \end{aligned} \tag{47}$$

Employ the identity

$$\delta(\lambda' - \lambda) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk e^{-ik(\lambda' - \lambda)/2\hbar}, \tag{48}$$

where k is the variable conjugated to λ as we stated in Eq. (17), we are led to the following expression for Eq. (44)

$$\begin{aligned} & \left\langle \vec{\lambda}^{n+1} \left| e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}^{grav}} \right| \vec{\lambda}^n \right\rangle \\ = \; & \frac{1}{(\pi)^3} \int d\vec{k}^n e^{-i\vec{k}^n \cdot (\vec{\lambda}^{n+1} - \vec{\lambda}^n)/2\hbar} \Big\{ 1 + \frac{i}{2\hbar} \epsilon\alpha\pi G \sqrt{v^n v^{n+1}} \\ & \times \Big[(v')^n \Big(e^{i\left(-k_1^n \lambda_1^n \frac{1}{v'^n} - k_2^n \lambda_2^n \frac{1}{v^n}\right)\ell_0} - e^{i\left(k_1^n \lambda_1^n \frac{1}{v'^n} - k_2^n \lambda_2^n \frac{1}{v^n}\right)\ell_0} \cdots + \mathit{cyclic\ terms} \Big) \\ & + (v'')^m \Big(e^{i\left(k_1^n \lambda_1^n \frac{1}{v''^n} + k_2^n \lambda_2^n \frac{1}{v^n}\right)\ell_0} - e^{i\left(-k_1^n \lambda_1^n \frac{1}{v''^n} + k_2^n \lambda_2^n \frac{1}{v^n}\right)\ell_0} \cdots + \mathit{cyclic\ terms}) \Big) + \mathcal{O}(\epsilon^2) \Big\}. \end{aligned} \tag{49}$$

Let $\ell_0 k_i^n \lambda_i^n / v^n = a_i$, $\ell_0 k_i^n \lambda_i^n / (v')^n = a'_i$, and $\ell_0 k_i^n \lambda_i^n / (v'')^n = a''_i$ for short. The exponential terms of this equation can be expressed as

$$\begin{aligned} & v' \left\{ [\cos(a'_1 + a_2) - i\sin(a'_1 + a_2) - \cos(a'_1 - a_2) - i\sin(a'_1 - a_2)] + \mathit{cyclic\ terms} \right\} \\ & + v'' \left\{ [\cos(a''_1 + a_2) + i\sin(a''_1 + a_2) - \cos(a''_1 - a_2) + i\sin(a''_1 - a_2)] + \mathit{cyclic\ terms} \right\}. \end{aligned} \tag{50}$$

Here, we introduce another approximation: assume the volume of the elementary cell is large enough, then we can let $v = v' = v''$, and therefore the $i\sin$ terms in Eq. (50) are canceled with each other. So we have

$$\begin{aligned} & \left\langle \vec{\lambda}^{n+1} \left| e^{\frac{i}{\hbar}\epsilon\alpha\hat{C}^{grav}} \right| \vec{\lambda}^n \right\rangle \\ = \; & \frac{1}{(\pi)^3} \int d\vec{k}^n e^{-i\vec{k}^n \cdot (\vec{\lambda}^{n+1} - \vec{\lambda}^n)/2\hbar} \Big\{ 1 + \frac{i}{\hbar} \epsilon\alpha \frac{\pi G}{2} \sqrt{v^n v^{n+1}} \, (-8v^n) \\ & \times \Big(\sin\frac{k_1^n \lambda_1^n}{v^n} \ell_0 \sin\frac{k_2^n \lambda_2^n}{v^n} \ell_0 + \sin\frac{k_2^n \lambda_2^n}{v^n} \ell_0 \sin\frac{k_3^n \lambda_3^n}{v^n} \ell_0 + \sin\frac{k_1^n \lambda_1^n}{v^n} \ell_0 \sin\frac{k_3^n \lambda_3^n}{v^n} \ell_0 + \mathcal{O}(\epsilon^2) \Big) \Big\}. \end{aligned} \tag{51}$$

Combining the scalar field part presented in Eq. (29), the extraction amplitude takes the form

$$\begin{aligned} A(\vec{\lambda}_f, \phi_f; \vec{\lambda}_i, \phi_i) &= \left\langle \vec{\lambda}^f, \phi^f \left| e^{\frac{i}{\hbar}\alpha \hat{C}^{grav}} \right| \vec{\lambda}^i, \phi^i \right\rangle \\ &= \sum_{\vec{\lambda}^{N-1}\ldots\vec{\lambda}^1} \frac{1}{(\pi)^{3N}} \int d\vec{k}_N \cdots d\vec{k}_1 \cdot \left(\frac{1}{2\pi}\right)^{2N} \int dp_N \cdots dp_1 e^{\frac{i}{\hbar}S_N} + \mathcal{O}(\epsilon^2), \end{aligned} \tag{52}$$

where

$$\begin{aligned} S_N &= \epsilon \sum_{n=0}^{N-1} \left\{ p^{n+1} \frac{\phi^{n+1} - \phi^n}{\epsilon} - \frac{\vec{k}^n}{2} \cdot \frac{(\vec{\lambda}^{n+1} - \vec{\lambda}^n)}{\epsilon} + \alpha \left[(p^n)^2 - 4\pi G \sqrt{v^n v^{n+1}} v^n \right.\right. \\ &\quad \left.\left. \times \left(\sin \frac{k_1^n \lambda_1^n}{v^n} \ell_0 \sin \frac{k_2^n \lambda_2^n}{v^n} \ell_0 + \sin \frac{k_2^n \lambda_2^n}{v^n} \ell_0 \sin \frac{k_3^n \lambda_3^n}{v^n} \ell_0 + \sin \frac{k_1^n \lambda_1^n}{v^n} \ell_0 \sin \frac{k_3^n \lambda_3^n}{v^n} \ell_0 \right) \right] \right\}. \end{aligned} \tag{53}$$

Finally, taking $N \to \infty$, we get

$$A\left(\vec{\lambda}^f, \phi^f; \vec{\lambda}^i, \phi^i\right) = \int d\alpha \int \left[D\vec{\lambda}(\tau)\right] \left[D\vec{k}(\tau)\right] \left[Dp(\tau)\right] \left[D\phi(\tau)\right] e^{\frac{i}{\hbar}S} \tag{54}$$

where

$$\begin{aligned} S &= \int_0^1 d\tau \left\{ p\dot{\phi} - \frac{1}{2}\vec{k} \cdot \dot{\vec{\lambda}} - \alpha \left[p^2 - 4\pi G \right.\right. \\ &\quad \times v^2 \left(\sin \frac{k_1 \lambda_1}{v} \ell_0 \sin \frac{k_2 \lambda_2}{v} \ell_0 + \sin \frac{k_2 \lambda_2}{v} \ell_0 \sin \frac{k_3 \lambda_3}{v} \ell_0 \right. \\ &\quad \left.\left.\left. + \sin \frac{k_1 \lambda_1}{v} \ell_0 \sin \frac{k_3 \lambda_3}{v} \ell_0 \right) \right] \right\}. \end{aligned} \tag{55}$$

Eq. (54) and Eq. (55) are the final expressions of the extraction amplitude and its action in $\bar{\mu}'$ scheme. Notice that λ_i has already been taken as a continuous variable, therefore, it is reasonable to interpret the $\left(\vec{\lambda}^{n+1} - \vec{\lambda}^n\right)/\epsilon$ as derivative here, and the integration ranges of $\vec{\lambda}$ and $\vec{k}$ are taken from $-\infty$ to ∞ directly. We no longer need to play the trick which was used in $\bar{\mu}$ scheme and the isotropic case to transform the integration range. Integration is over all trajectories in the classical phase space from the beginning.

V. DISCUSSION

In this paper we extended the phase space path integral formulation of Friedmann space-times to anisotropic Bianchi I models, and performed the calculation both in $\bar{\mu}$ and $\bar{\mu}'$ scheme. We restricted the matter source to be a massless scalar field and focused on the positive octant in which all three p_i are positive.

Since $\bar{\mu}_i = \sqrt{\Delta/p_i}$ in $\bar{\mu}$ scheme, by setting $\nu_i \sim p_i^{3/2}$, just as $\nu \sim p^{3/2}$ in the isotropic case, the procedure used in [9] could be implemented directly. The formulation of the extraction amplitude then resembles three copies of the isotropic model as we desired. To incorporate the $\bar{\mu}'$ scheme, we have to overcome a few obstacles. The main problem comes from the fact that the expression of the $\bar{\mu}'_i$, which was given by Eq. (15), is much more complicated in $\bar{\mu}'$ scheme than in $\bar{\mu}$ scheme. This leads to the consequence that we have to make an algebraic simplification by introducing $\lambda_i \sim p_i^{1/2}$, and then the wave function is dragged along the λ_1 direction by the unitary shift operator E_1. However, the affine distance involved in this dagging depends on λ_2, λ_3. From the difference equation of the gravitational part of the total constraint, it can be seen that the new variable λ_i has a discrete spectrum but does not support on a specific λ_i-lattice. We can not write an integral representation for the Kronecker delta by using a standard residue calculation like Eq. (32).

In order to perform a path integral formulation in $\bar{\mu}'_i$ case, we have used two approximations in IV. The first one is taking λ_i as continuous variable, then one can use the Dirac delta to express the inner product. The second one is to presume the volume of elementary cell is large enough such that $(v-2) \approx v \approx (v+2)$. Actually they are the same thing. In general, our treatment presupposed that the the calculation is away from the Planck regime, in other words, a semiclassical approximation was taken. This is the key point where $\bar{\mu}'$ is different from $\bar{\mu}$ and the isotropic case. In the two later situations the calculation is exact and does not need any additional assumption.

If we compare the two path integrals in $\bar{\mu}$ and $\bar{\mu}'$ scheme, it is obviously that the actions have the same formulation as in isotropic case: a classical action plus a quantum gravity correction term, i.e., the sin term. Ignoring the constant terms, from Eq. (43) and Eq. (55), one can write down directly the two quantum gravity correction terms in the two schemes (denoted by T and T' respectively). In $\bar{\mu}$ scheme,

$$\begin{aligned} T &= \nu^1\nu^2 \sin\ell_0 b^1 \sin\ell_0 b^2 + \nu^2\nu^3 \sin\ell_0 b^2 \sin\ell_0 b^3 \\ &\quad + \nu^1\nu^3 \sin\ell_0 b^1 \sin\ell_0 b^3. \end{aligned} \tag{56}$$

In $\bar{\mu}'$ scheme,

$$\begin{aligned} T' &= v^2 \left(\sin\frac{k_1\lambda_1}{v}\ell_0 \sin\frac{k_2\lambda_2}{v}\ell_0 + \sin\frac{k_2\lambda_2}{v}\ell_0 \sin\frac{k_3\lambda_3}{v}\ell_0 \right. \\ &\quad \left. + \sin\frac{k_1\lambda_1}{v}\ell_0 \sin\frac{k_3\lambda_3}{v}\ell_0 \right). \end{aligned} \tag{57}$$

These terms do not look the same. By using the explicit expressions of b_i, λ_i and k_i, T and T' could be expressed as

$$T = \nu^1\nu^2 \sin\bar{\mu}_1 c_1 \sin\bar{\mu}_2 c_2 + \nu^2\nu^3 \sin\bar{\mu}_2 c_2 \sin\bar{\mu}_3 c_3 + \nu^1\nu^3 \sin\bar{\mu}_1 c_1 \sin\bar{\mu}_3 c_3, \tag{58}$$

and

$$T' = v^2 \left(\sin\bar{\mu}_1' c_1 \sin\bar{\mu}_2' c_2 + \sin\bar{\mu}_2' c_2 \sin\bar{\mu}_3' c_3 + \sin\bar{\mu}_1' c_1 \sin\bar{\mu}_3' c_3 \right). \tag{59}$$

Notice that $\nu_i \sim p_i^{3/2}$ and $v \sim (p_1 p_2 p_3)^{1/2}$. Now we find that T and T' are extremely similar in the formulation. The difference is the immediate cause for the fiducial cell dependence on the semiclassical limit in $\bar{\mu}$ scheme. Consider the fiducial cell $\mathcal{V}$. Its volume is $V_0 = L_1L_2L_3$. Notice that $c_i \sim L_i\dot{a}_i$ and $p_i \sim L_jL_ka_ja_k$, so if we rescale $\mathcal{V}$ as

$$V_0 = L_1L_2L_3 \to V_0' = l_1L_1l_2L_2l_3L_3 = l_1l_2l_3V_0, \tag{60}$$

then

$$\bar{\mu}_1 c_1 \to \frac{l_1}{\sqrt{l_2 l_3}}\bar{\mu}_1 c_1, \tag{61}$$

and

$$\bar{\mu}_1' c_1 \to \bar{\mu}_1' c_1. \tag{62}$$

This is the reason why $\bar{\mu}'$ scheme has better scaling properties. These properties are already known in canonical quantization program. In path integral formulation, we have proved again that the quantum dynamic does depend on the choice of the fiducial cell in $\bar{\mu}$ scheme, while it does not in $\bar{\mu}'$ scheme. Furthermore, if we take $p_1 = p_2 = p_3$, and therefore $\nu_i = v$, then we find $T = T' \sim v^2 \sin^2\bar{\mu}c$. This is precisely the quantum correction term in Friedmann model. Hence, it is rational to forecast that in the path integral formulation of Bianchi I models the quantum bounce will also replace the big bang singularity due to the quantum gravity correction term. All of these properties provide a strong evidence for the equivalence of the canonical approach and the path integral approach in LQC.

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